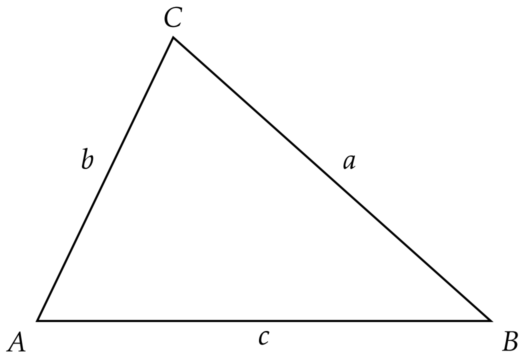
The Sine Rule

Labelling Convention



Capital letters for angles, lowercase for the side **opposite** that angle. So side a is opposite angle A.

Fact — **The Sine Rule:**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use this when you have a **matching pair** — a side and the angle opposite it.

Example

In $\triangle ABC$, $a = 7$ cm, $A = 58°$, $B = 23°$. Find the length b.

$$\frac{7}{\sin 58°} = \frac{b}{\sin 23°}$$
$$b = \frac{7 \times \sin 23°}{\sin 58°} = 3.23\ cm\ (3\ s.f.)$$

Example

In $\triangle ABC$, $a = 7$ cm, $A = 128°$, $b = 6$ cm. Find angle B.

Using the sine rule "upside down" to find an angle:

$$\frac{\sin B}{6} = \frac{\sin 128°}{7}$$

$$\sin B = \frac{6 \times \sin 128°}{7} = 0.6754\ldots$$

$$B = \sin^{-1}(0.6754) = 42.5° \text{ (3 s.f.)}$$

Note: $180° - 42.5° = 137.5°$ is impossible since $A = 128°$ is already obtuse.

Example

In $\triangle ABC$, $a = 5$ cm, $A = 72°$, $C = 41°$. Find the length c.

$$\frac{c}{\sin 41°} = \frac{5}{\sin 72°}$$

$$c = \frac{5 \times \sin 41°}{\sin 72°} = 3.45 \text{ cm (3 s.f.)}$$

The Cosine Rule

Example

In $\triangle ABC$, $b = 8$ cm, $c = 5$ cm, $A = 60°$. Can you use the sine rule? Why or why not?

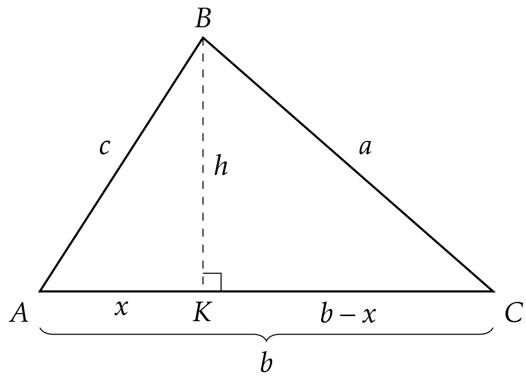
*No — we don't have a "matching pair". We have two sides and the **included** angle. We need the cosine rule.*

Fact — The Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Use this when you have **two sides and the included angle** (SAS), or **all three sides** (SSS).

Proof



Drop a perpendicular from B to AC, meeting at K. Let BK = h and AK = x.

In $\triangle ABK$: $h^2 = c^2 - x^2$

In $\triangle BCK$: $h^2 = a^2 - (b - x)^2$

Setting equal: $c^2 - x^2 = a^2 - (b - x)^2$

Expanding: $c^2 - x^2 = a^2 - b^2 + 2bx - x^2$

So: $a^2 = b^2 + c^2 - 2bx$

But $x = c\cos A$, *giving* $a^2 = b^2 + c^2 - 2bc\cos A$. □

Finding a side (SAS)

Example

In $\triangle ABC$, $b = 9$ cm, $a = 8$ cm, $C = 127°$. Find c.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 8^2 + 9^2 - 2(8)(9)\cos 127° \\ &= 64 + 81 - 144 \times (-0.6018\ldots) \\ &= 145 + 86.66\ldots = 231.66\ldots \end{aligned}$$

$c = \sqrt{231.66} = 15.2$ *cm (3 s.f.)*
NB: *Don't forget to take the square root!*

Example

In $\triangle PQR$, $p = 11$ cm, $r = 7$ cm, $Q = 48°$. Find q.

$$\begin{aligned} q^2 &= p^2 + r^2 - 2pr\cos Q \\ &= 11^2 + 7^2 - 2(11)(7)\cos 48° \\ &= 121 + 49 - 154 \times 0.6691\ldots \\ &= 170 - 103.04\ldots = 66.96\ldots \end{aligned}$$

$q = \sqrt{66.96} = 8.18$ *cm (3 s.f.)*

Finding an angle (SSS)

Example

In $\triangle ABC$, $a = 8$, $b = 9$, $c = 10$. Find the largest angle.

The largest angle is opposite the longest side, so we find C:

$c^2 = a^2 + b^2 - 2ab\cos C$, *so* $10^2 = 8^2 + 9^2 - 2(8)(9)\cos C$

$100 = 145 - 144\cos C$

$\cos C = \dfrac{145 - 100}{144} = \dfrac{45}{144} = 0.3125$

$C = \cos^{-1}(0.3125) = 71.8°$ *(3 s.f.)*

Example

In $\triangle ABC$, $a = 5$, $b = 6$, $c = 9$. Find angle C.

$c^2 = a^2 + b^2 - 2ab\cos C$, *so* $9^2 = 5^2 + 6^2 - 2(5)(6)\cos C$

$81 = 61 - 60\cos C$

$\cos C = \dfrac{61 - 81}{60} = \dfrac{-20}{60} = -\frac{1}{3}$

$C = \cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ *(1 d.p.)*

Note: $\cos C < 0$, *so C is obtuse — this makes sense since c is much larger than a and b.*

Area of a Triangle: $\frac{1}{2}ab\sin C$

Example

You know two sides and the included angle of a triangle. Can you find its area without finding the height first?

Using the same diagram as the cosine rule proof: the height $h = a\sin C$.
Area $= \frac{1}{2} \times base \times height = \frac{1}{2} \times b \times a\sin C = \frac{1}{2}ab\sin C$. □

Example

Find the area of $\triangle ABC$ where $a = 7$ cm, $b = 8.5$ cm, $C = 72°$.

$$Area = \frac{1}{2}(7)(8.5)\sin 72° = \frac{1}{2} \times 59.5 \times 0.9511 = 28.3\ cm^2\ (3\ s.f.)$$

Example

Find the area of $\triangle ABC$ where $a = 12.5$ cm, $A = 48°$, $B = 65°$.

First find $C = 180° - 48° - 65° = 67°$.
Use the sine rule to find b:

$$\frac{b}{\sin 65°} = \frac{12.5}{\sin 48°} \implies b = \frac{12.5 \sin 65°}{\sin 48°} = 15.244\ldots\ cm$$

$$Area = \frac{1}{2}(12.5)(15.244\ldots)\sin 67° = 87.7\ cm^2\ (3\ s.f.)$$

NB: *Keep the unrounded value of b on your calculator for the area calculation.*

Choosing the Right Rule

Fact — **Decision flowchart:**

1. Do you have a **matching pair** (side + opposite angle)?
   - Yes → **Sine Rule**
2. Do you have **two sides + included angle** (SAS)?
   - Yes → **Cosine Rule** (to find the third side)
3. Do you have **three sides** (SSS)?
   - Yes → **Cosine Rule** (to find an angle)
4. Always draw a diagram!
5. You may need to use $A + B + C = 180°$.

The Ambiguous Case of the Sine Rule

Example

In $\triangle ABC$, $a = 10$ cm, $b = 7$ cm, $A = 40°$. Find angle B.

$$\frac{\sin B}{7} = \frac{\sin 40°}{10} \implies \sin B = \frac{7 \sin 40°}{10} = 0.4499\ldots$$

$B = \sin^{-1}(0.4499) = 26.7°$ *(3 s.f.)*
But $\sin B = 0.4499$ *also gives* $B = 180° - 26.7° = 153.3°$.
Check: $A + B = 40° + 153.3° = 193.3° > 180°$ *— impossible!*
So only $B = 26.7°$ *works here.*

Example

In $\triangle ABC$, $a = 5$ cm, $b = 8$ cm, $A = 30°$. Find angle B.

$$\frac{\sin B}{8} = \frac{\sin 30°}{5} \implies \sin B = \frac{8 \times 0.5}{5} = 0.8$$

$B = \sin^{-1}(0.8) = 53.1°$ *or* $B = 180° - 53.1° = 126.9°$
Check: $A + B = 30° + 53.1° = 83.1° < 180°$ ✓
Check: $A + B = 30° + 126.9° = 156.9° < 180°$ ✓
Both *values are valid — there are* ***two possible triangles****.*
This is the ***ambiguous case****: it arises when you are given two sides and an angle* ***not*** *between them (SSA).*

Fact — The ambiguous case occurs when using the sine rule to find an angle. Since $\sin\theta = \sin(180° - \theta)$, there may be **two valid solutions**. Always check whether the second solution gives a valid triangle (angles summing to less than 180°).